Math 10B with Professor Stankova
Worksheet, Discussion #20; **Thursday, 4/5/2018**
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1 Second-Order Linear ODEs

1.1 Concepts

1. Similar to solving second order linear recurrence relations, we guess the solution is of the form λ^n but in this case, we let $\lambda = e^r$ and $n = t$ so we guess the solution is of the form e^{rt}. Then for a ODE $ay'' + by' + cy = 0$, we get the characteristic equation $ar^2 + br + c = 0$ and solve for the roots and the solution is of the form $c_1e^{r_1t} + c_2e^{r_2t}$. If the roots are repeated, the solution is of the form $c_1e^{rt} + c_2te^{rt}$. If the roots are complex of the form $a \pm bi$, then the solution is of the form $c_1e^{at}\sin(bt) + c_2e^{at}\cos(bt)$.

   A **initial value problem** for second linear recurrence relation will tell you $y(0)$ and $y'(0)$. A **boundary value problem** will tell you $y(a), y(b)$, where a, b are the boundaries of the interval you want to find the solution on. An initial value problem will always have a solution. But, a boundary value problem may not have a solution, a unique solution, or infinitely many solutions. It can never have 2, 3, or any finite number other than 0 or 1 solutions.

1.2 Examples

2. Solve the initial value problem $2y'' + 4y' + 2y = 0$ with $y(0) = 0, y'(0) = 1$.
3. Solve the boundary value problem of a mass on a spring given by $x'' = -4x$ and $x(0) = 0, x(\pi) = 0$.

1.3 Problems

4. True False It is possible for there to be no solution to an initial value problem.
5. True False It is possible for there to be no solution to a boundary value problem.
6. True False All linear ODEs have the property that linear combinations of their solutions are also solutions to them.
7. Solve the initial value problem given by $3y'' = 15y' - 18y$ and $y(0) = 0$ and $y'(0) = 1$.
8. Solve the boundary value problem given by $y'' = -y$ and $y(0) = 0, y(\pi) = 1$.
9. Find the second order linear ODE such that $y(t) = e^{2t}\sin(t)$ is a solution to it.
10. What is the smallest value of $\alpha > 0$ such that any solution of $y'' + \alpha y' + y = 0$ does not oscillate (does not have any terms of $\sin, \cos$).

1.4 Extra Problems

11. Solve the initial value problem $3y'' + 18y' + 27y = 0$ with $y(0) = 0, y'(0) = 1$.

12. Solve the initial value problem given by $2y'' = 3y' - y$ and $y(0) = 0$ and $y'(0) = 1$.

13. Solve the boundary value problem given by $y'' + 2y' + 5y = 0$ and $y(0) = 0, y(1) = 1$.

14. Find the second order linear ODE such that $y(t) = te^{2t}$ is a solution to it.

15. What is the largest value of $\alpha > 0$ such that any solution of $y'' + 4y' + \alpha y = 0$ does not oscillate (does not have any terms of $\sin, \cos$).

2 Euler's Method

2.1 Concepts

16. Euler's method allows us to approximate solutions to differential equations. Given a differential equation $y' = f(y,t)$ and an initial condition $y(0) = y_0$ and a step size h, we can approximate the path by $y_{n+1} = y_n + f(y_n, t_n)h$. This is gotten by writing $y' = \frac{dy}{dt} \approx \frac{y_{n+1} - y_n}{h}$.

    A slope field is a graph where at every point y, t, you draw a line with the slope there, which is given by the function $f(y,t)$.

2.2 Examples

17. Consider the differential equation $y' = x - y^2$ with initial condition $y(0) = 1$. Use Euler's method to approximate $y(3)$ using step sizes of 1.

2.3 Problems

18. True False Autonomous equations like $y' = 2\sqrt{y}$ will have slope field that are the same after shifting left and right.

19. True False We can only use slope fields and Euler's method when we are given a first order equation.

20. Draw a slope field for $y' = y^2 + x^2$ and sketch the solution when $y(0) = 0$ on the interval $-2 \leq x \leq 2, -2 \leq y \leq 2$.

21. Use Euler's method to estimate $y(3)$ given that $y' = x^2 + y^2$ and $y(0) = 0$ using step sizes of 1.

22. Draw a slope field for $y' = y^2 - x^2$ and sketch the solution when $y(0) = 1$ on the interval $0 \leq x \leq 4, 0 \leq y \leq 4$.

23. Use Euler's method to estimate $y(3)$ given that $y' = y^2 - x^2$ and $y(0) = 1$ using step sizes of 1.